ON THE SCATTERING OF RADIATION IN A MEDIUM WITH CHANGING OPTICAL PROPERTIES

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This paper gives a solution of the problem of the variable scattering and absorption of light in a medium, the optical properties of which are changing. The amount of scattering material (for example, droplets of water existing in the dispersed state in air) is assumed to depend on the total energy of absorbed light. The initial concentration of the scattering material is assumed to be constant. At later times, it decreases, the decrease depending linearly on the quantity of absorbed radiation.

Relations of this type will be observed, for example, in the case where solar radiation acts on a cloud, when the quantity of moisture occurring in the form of an aerosol will be reduced by vaporization. The rate of vaporization – that is, the quantity of water vaporized in unit time – will obviously be proportional to the quantity of absorbed radiation. Any inertia of this process is neglected.

We shall be concerned with a one-dimensional problem: that is, with a semi-infinite medium which consists of parallel layers, the optical properties of which are functions of z and t. Radiation from outside this region penetrates into the medium.

1. We propose for consideration a radiation flux I which is a function of the time t, the coordinates z and, in the general case, of two angular coordinates θ and ϑ, the center of which is located at the point from which the radiation originates.

In this case we have for the determination of I a system of two equations [1]

$$\frac{dI}{ds} = -\alpha^* I + \varepsilon, \quad \varepsilon = \beta\alpha^* \iint_{\Omega} I \frac{d\omega'}{4\pi} \tag{1.1}$$

Here the case is considered in which equilibrium exists at each moment of time, as far as radiation is concerned; and therefore in Equation (1.1) one may neglect the time derivative of I which is divided by the speed of light [2]. In the second expression, the integration is carried out over the solid angle. We assume the scattering is uniform over a sphere. In the more general case, the integral will have the following form:

$$\iint_{\Omega} I\varkappa(\gamma) \frac{d\omega'}{4\pi}$$

Here $\kappa(\gamma)$ is the scattering index, which specifies the fraction of radiation flux scattered in a given direction. The quantity γ is the angle between the incident and scattered light.

In the case of the spherical index for light-scattering, the light is scattered equally in all directions, so that $\kappa(\gamma) = 1$.

In Equations (1.1), a^* is the volume coefficient of absorption of the medium, that is, the quantity of radiation (expressed, for example, in calories), which is absorbed per unit volume.

We shall consider the medium in which the scattering material is contained; the volume coefficient of absorption will be proportional to the concentration of that material $\rho(z, t)$. Thus, $a^* = a\rho(z, t)$.

The quantity β (the ratio of the scattered to the absorbed radiation) will be assumed constant.

Thus the basic assumptions relating to absorption and scattering are as follows:

a) The absorption coefficient is proportional to the concentration of the absorbing material. This would be valid, for example, in the case where absorption is dependent on multiple reflection and refraction from the different particles of which the absorbing and scattering material is made.

b) The ratio of scattered radiation to absorbed radiation is constant, and does not depend on the dispersion of the particles. This corresponds to the assumption that the albedo of each particle does not depend on its size.

Equation (1.1) may be written in the following form:

$$\cos\vartheta\frac{\partial I}{\partial z} = -\alpha\rho(z, t)\,I + \varepsilon, \quad \varepsilon = \beta\alpha\rho(z, t)\iint\limits_{\Omega} I\,\frac{d\omega'}{4\pi} \tag{1.2}$$

Or, in other words

$$\cos\vartheta\frac{\partial I}{\partial z}\frac{1}{\alpha\rho(z, t)} = -I + \beta\iint\limits_{\Omega} I\,\frac{d\omega'}{4\pi} \tag{1.3}$$

where ϑ is the angle between the direction of the beam of radiation and the z-axis.

If we introduce the new variable

$$\tau = \alpha\int\limits_0^z \rho(\zeta, t)\,d\zeta \tag{1.4}$$

then from (1.3) we obtain

$$\cos\vartheta\,\frac{\partial I}{\partial\tau} = -I + \beta\iint\limits_{\Omega} I\,\frac{d\omega'}{4\pi} \tag{1.5}$$

Equation (1.5) corresponds to the case of absorption and scattering of radiation in a medium for which the coefficient of absorption is constant and equal to unity, and in which the variable τ performs the function of the variable z.

We first consider the case where the intensity of radiation entering the medium from outside is constant. Later we shall also consider the case where the intensity of radiation depends on time. If we introduce the function $B(\tau)$, equal to the quantity of radiation absorbed per unit volume and per unit time

$$B(\tau) = \iint\limits_{\Omega} I\,d\omega' \tag{1.6}$$

then one obtains the following integral equation [1] for its determination:

$$B(\tau) = I_0 e^{-\tau} + \frac{\beta}{2}\int\limits_0^\infty B(\omega)\,\mathrm{Ei}(|\tau-\omega|)\,d\omega \tag{1.7}$$

Here

$$\mathrm{Ei}(s) = \int\limits_s^\infty \frac{e^{-x}}{x}\,dx \tag{1.8}$$

In calculating $B(\tau)$ it is convenient to use the method of successive approximations. The first approximation, corresponding to the case of the absence of scattering, will be

$$B_1(\tau) = I_0 e^{-\tau} \tag{1.9}$$

The second approximation

$$B_2(\tau) = I_0\left(e^{-\tau} + \frac{\beta}{2}\int_0^\infty e^{-\omega}\,\mathrm{Ei}\,(|\tau - \omega|)\,d\omega\right) \tag{1.10}$$

Succeeding approximations may be obtained in an analogous manner.

If we revert to the original system of coordinates, then for the amount of absorption of radiation per unit volume we have the expression

$$B(z, t) = B(\tau)\frac{\partial \tau}{\partial z} \tag{1.11}$$

Since, according to the original assumptions, the concentration of the scattering material decreases linearly with the quantity of absorbed radiation, we obtain for determination of $\rho(z, t)$, the following equation:

$$\rho(z, t) = \rho_0 - \lambda\int_0^t B(\tau)\frac{\partial \tau}{\partial z}\,d\xi \tag{1.12}$$

Here ρ_0 is the initial concentration of scattering material, and λ is the coefficient of proportionality relating the rate of change of concentration of that material with the intensity of absorbed radiation.

From (1.4) we have

$$\rho(z, t) = \frac{1}{\alpha}\frac{\partial \tau}{\partial z} \tag{1.13}$$

On the basis of (1.12) and (1.13) we obtain

$$\frac{\partial \rho(z, t)}{\partial t} = -\lambda B(\tau)\frac{\partial \tau}{\partial z}, \quad \text{or} \quad \frac{\partial^2 \tau}{\partial z\,\partial t} = -\lambda\alpha B(\tau)\frac{\partial \tau}{\partial z} \tag{1.14}$$

We introduce the function

$$\Phi(\tau) = \int_\tau^\infty B(\omega)\,d\omega \tag{1.15}$$

Then we obtain

$$B(\tau)\frac{\partial \tau}{\partial z} = -\frac{\partial \Phi(\tau)}{\partial z} \tag{1.16}$$

If we consider successive approximations, then, using (1.9), we obtain

$$\Phi_1(\tau) = I_0\int_\tau^\infty e^{-\omega}\,d\omega = I_0 e^{-\tau} \tag{1.17}$$

The next approximation, resulting from (1.10), will be

$$\Phi_2(\tau) = I_0\Big(e^{-\tau} + \frac{\beta}{2}\int_{\tau}^{\infty} d\tau \Big(\int_0^{\infty} e^{-\omega}\,\mathrm{Ei}\,(|r-\omega|)\,d\omega\Big)\Big) \tag{1.18}$$

On the basis of (1.16), Equation (1.14) takes the following form:

$$\frac{\partial^2\tau}{\partial z\partial t} = \lambda\alpha\frac{\partial\Phi(\tau)}{\partial z}, \qquad \text{or} \qquad \frac{\partial}{\partial z}\Big[\frac{\partial\tau}{\partial t} - \mu\Phi(\tau)\Big] = 0 \qquad (\mu = \lambda\alpha) \tag{1.19}$$

The function τ, which satisfies this equation, may be expressed through functions of z and t.

From (1.19) it follows that τ satisfies the following ordinary differential equation:

$$\frac{d\tau}{dt} - \mu\Phi'(\tau) = f(t) \tag{1.20}$$

We set the boundary conditions for the determination of τ. The function τ is determined within the quandrant of the zt surface in which $0 < t < \infty$ and $0 < z < \infty$. We have the expression (1.4)

$$\tau = \alpha\int_0^z \rho(\zeta, t)\,d\zeta$$

From this it follows that $\tau = 0$ where $z = 0$. At $t = 0$, the initial condition is $\rho(z, t) = \rho_0$ On this basis, $\tau = \tau_0 = \alpha\rho_0 z$ for $t = 0$. Since $\tau = 0$ for $z = 0$, and in view of the fact that $d\tau/dt = 0$ from (1.20), we have

$$f(t) = -\mu\Phi(0) \tag{1.21}$$

From this we obtain

$$\frac{d\tau}{dt} = \mu(\Phi(\tau) - \Phi(0)), \quad \text{or} \quad \int_{\tau_0}^{\tau}\frac{d\xi}{\Phi(\xi) - \Phi(0)} = \mu t \tag{1.22}$$

Since $\tau_0 = \alpha\rho_0 z$ for $t = 0$, we finally obtain

$$t = \frac{1}{\mu}\int_{\alpha\rho_0 z}^{\tau}\frac{d\xi}{\Phi(\xi) - \Phi(0)} \tag{1.23}$$

From this, after some manipulation, and determining τ as a function of z and t from (1.13), we obtain the concentration of the scattering material $\rho(z, t)$. We obtain the function $\Phi(\tau)$ from the integral equation (1.7) and the relation (1.15).

Approximate expressions for $\Phi(\tau)$ are given in (1.17) and (1.18).

In the case considered above, the intensity of incoming radiation was constant. If this quantity (designated by $I_0(t)$) depends on time, then from (1.12) we have

$$\rho(z, t) = \rho_0 - \lambda \int_0^t I_0(s) B(\tau) \frac{\partial \tau}{\partial z} ds \tag{1.24}$$

In this case it is necessary to introduce a new variable

$$\sigma = \int_0^t I_0(s) ds \tag{1.25}$$

as a result of which Expression (1.24) is transformed into the form (1.12). Thereupon, repeating the previous steps, we arrive at the following equation for the determination of τ:

$$\frac{1}{\lambda\alpha} \int_{\alpha\rho_0 z}^{\cdot} \frac{d\xi}{\Phi(\xi) - \Phi(0)} = \int_0^t I_0(s) ds \tag{1.26}$$

2. We consider the problem in which radiation is only absorbed and not scattered. This would correspond to the case where we use Expression (1.17) to specify $\Phi(\tau)$, which appears in Equation (1.19). In this case it is not hard to obtain an expression for $\rho(z, t)$ in explicit form. It can be obtained by transformation of the integral (1.23), which in this case is reduced to elementary functions. A slightly different method of obtaining τ and ρ is given below.

Thus, let $\Phi(\tau) = I_0 e^{-\tau}$. In this case the equation takes the form

$$\frac{d\tau}{dt} - \mu I_0 e^{-\tau} = f(t) \tag{2.1}$$

The solution of this equation depends on one function of z, and also on one function of t, since the function $f(t)$ on the right is arbitrary. The substitution $\phi = e^{-\tau}$ reduces (2.1) to the Bernoulli equation

$$\frac{d}{dt}(\lg\varphi) + \nu\varphi = -f(t), \quad \text{or} \quad \frac{d\varphi}{dt} + f(t)\varphi + \nu\varphi^2 = 0 \qquad (\nu = \mu I_0) \tag{2.2}$$

We introduce a new variable $s = \nu t$ and designate

$$g^*(s) = \frac{1}{\nu} f\left(\frac{t}{\nu}\right), \qquad \psi = \exp\int_0^s \varphi d\xi \tag{2.3}$$

For the function ψ we obtain the following expression:

$$\psi = m^*(z) \exp\left(-\int_0^s g^*(\xi) d\xi\right) + h(z) \tag{2.4}$$

From this we obtain

$$\varphi = \frac{d}{ds}[\lg(g(s) + h(z))] \qquad \left(g(s) = \exp\int_0^s g^*(\xi)\,d\xi\right) \tag{2.5}$$

Thereupon we obtain an expression for ϕ

$$\varphi = \frac{g'(s)}{g(s) + h(z)} \tag{2.6}$$

Further, in conformity with (1.13) and (2.3), we have

$$\rho(z, t) = -\frac{1}{\alpha}\frac{d}{dz}[\lg\varphi] \tag{2.7}$$

Substituting in this Expression (2.6), we obtain

$$\rho(z, t) = \frac{1}{\alpha}\frac{h'(z)}{g(s) + h(z)} \tag{2.8}$$

To determine the functions $g(s)$ and $h(z)$ we use the boundary conditions. Since $r = -\lg\phi = 0$ for $z = 0$, and consequently $\phi = 1$, then

$$\varphi = \frac{g_0 e^s}{g_0 e^s - h(0) + h(z)} \tag{2.9}$$

On the basis of the condition that $\rho = \rho_0$ for $t = 0$ or $s = 0$, we obtain

$$h(z) = g_0 e^{\alpha\rho_0 z} - g_0 \tag{2.10}$$

Using (2.10) and (2.11), and also reverting to the previous variables, we obtain for the concentration of scattering material

$$\rho(z, t) = \rho_0 \frac{e^{\alpha\rho_0 z}}{e^{\alpha\lambda I_0 t} - 1 + e^{\alpha\rho_0 z}} \tag{2.11}$$

Expression (2.12) was obtained for the case in which the radiation begins to penetrate the medium at the time $t = 0$, and is subsequently constant in intensity. However, the result can easily be generalized to time-dependent radiation by the method reported at the end of the preceding section.

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